THE INFLUENCE OF ANISOTROPIC CONDUCTIVITY ON THE UNSTEADY MOTION OF A CONDUCTING GAS IN A PLANE CHANNEL

(VLIIANIE ANIZOTROPII PROVODIMOSTI NA NEUSTANOVIVSHEESIA DVIZHENIE PROVODIASHCHEGO GAZA V PLOSKOM KANALE)

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In this paper we study the unsteady flow of a weakly ionized inviscid gas between parallel walls in the presence of a transverse magnetic field.

By means of a Laplace transformation we obtain the exact solution of the problem under the assumption that we can neglect the drift of ions relative to the gas, and also the effects of compressibility. Reduction of the solution to real form is achieved for the case of ideally conducting walls.

It is shown that in contrast to the usual case of isotropic conductivity the nonstationary regime under consideration always has the nature of damped oscillations.

1. Formulation of the problem. Let us consider the unsteady motion of a gas which conducts electricity between two parallel plates (the walls of the channel), subjected to a constant pressure gradient P_x and with a uniform magnetic field perpendicular to the walls (Fig. 1).

If $\omega\tau \ll 1$, then the conductivity can be taken to be a scalar quantity (ω is the cyclotron frequency of the charged particles, τ is the mean time between collisions). Under these conditions in the problem under consideration not only the velocity but also the induced magnetic field has only a component in the direction of the imposed pressure gradient (Ox), whilst the electric field and the current are directed along the axis of y. If, however, the condition $\omega\tau \ll 1$ is violated, then the conductivity will have an anisotropic character and the flow of gas will be complicated, since the vector of current density will acquire a component along the x-axis, causing in its turn a cross flow of the gas, and so on.

In what follows we shall assume that the relation $\omega_i \tau_i \ll 1$ is fulfilled for the ions. This enables us to neglect the effect of the drift of the ions relative to the gas and to make use of Ohm's law in the form [1-3]

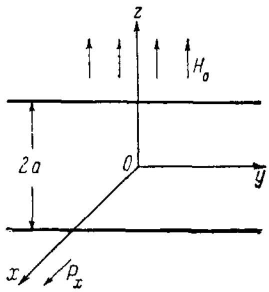

Fig. 1.

$$\mathbf{j} + \frac{\omega_e \tau^*}{H} \mathbf{j}\times\mathbf{H} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{H}) \tag{1.1}$$

where $\mathbf{j}$ is the current density, $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields, $\mathbf{v}$ is the velocity of the gas, σ is the conductivity, ω_e is the cyclotron frequency of the electrons, τ^* is the mean time between collisions of electrons with ions and neutral particles (assuming the *CGSM* system, where the coefficient of magnetic permeability is taken as equal to unity).

The system of equations of magnetohydrodynamics for an ideal incompressible medium takes the form (ρ is the density, p is the pressure of the gas)

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = -\nabla p + \mathbf{j}\times\mathbf{H}, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{H} = 0, \qquad \mathbf{j} = \frac{1}{4\pi}\operatorname{rot}\mathbf{H}$$

In the case under consideration the velocity vector and the induced magnetic field have components only along the axes of x and y, which depend upon z and t and satisfy a system of partial differential equations. After introducing the dimensionless quantities

$$\mathbf{u} = \frac{\mathbf{v}}{v_0}, \quad \mathbf{h} = \frac{\mathbf{H}}{H_0}, \quad \mathbf{e} = \frac{\mathbf{E}}{v_0 H_0}, \quad \zeta = \frac{z}{a}, \quad \tau = \frac{v_0 t}{a}$$
$$P = \frac{P_x a}{\rho v_0^2}, \quad R_m = 4\pi\sigma a v_0, \quad S = \frac{H_0^2 \sigma a}{\rho v_0}, \quad \beta = \omega_e \tau^* \frac{H_0}{H} = \frac{e}{m} H_0 \tau^* \tag{1.3}$$

(v_0 is a characteristic velocity, e and m are the charge and mass of the electron) the aforesaid system takes the form

$$\frac{\partial h_x}{\partial \tau} = \frac{1}{R_m}\left(\frac{\partial^2 h_x}{\partial \zeta^2} + \beta\frac{\partial^2 h_y}{\partial \zeta^2}\right) + \frac{\partial u_x}{\partial \zeta}, \qquad \frac{\partial h_y}{\partial \tau} = \frac{1}{R_m}\left(\frac{\partial^2 h_y}{\partial \zeta^2} - \beta\frac{\partial^2 h_x}{\partial \zeta^2}\right) + \frac{\partial u_y}{\partial \zeta}$$
$$\frac{\partial u_x}{\partial \tau} = P + \frac{S}{R_m}\frac{\partial h_x}{\partial \zeta}, \qquad \frac{\partial u_y}{\partial \tau} = \frac{S}{R_m}\frac{\partial h_y}{\partial \zeta} \tag{1.4}$$

If we define the components of pressure gradient by the relations

$$\frac{\partial p}{\partial x} = -P_x, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\frac{1}{8\pi}\frac{\partial H^2}{\partial z} \tag{1.5}$$

then all the equations of magnetohydrodynamics are satisfied when the components of the electric field are given by the following formulas

$$e_x = \frac{1}{R_m}\left(\beta\frac{\partial h_x}{\partial \zeta} - \frac{\partial h_y}{\partial \zeta}\right) - u_y, \quad e_y = \frac{1}{R_m}\left(\beta\frac{\partial h_y}{\partial \zeta} + \frac{\partial h_x}{\partial \zeta}\right) + u_x, \quad e_z = u_y h_x - u_x h_y - \frac{\beta}{2R_m}\frac{\partial h^2}{\partial \zeta} \tag{1.6}$$

The system (1.4) has to be solved from zero initial conditions

$$u_x = u_y = h_x = h_y = 0 \quad \text{when } \tau = 0 \tag{1.7}$$

and boundary conditions which ensure continuity of tangential components of electric and magnetic fields on passing from the region of the gas to the walls of the channel (the index * relates to the region of the walls)

$$h_x = h_x^*, \quad h_y = h_y^*, \quad e_x = e_x^*, \quad e_y = e_y^* \quad \text{when } \zeta = \pm 1 \tag{1.8}$$

Accordingly, to solve the problem in the case of walls of finite conductivity (σ^*) it is necessary along with the system (1.4) to consider the equations of electrodynamics in the region $|\zeta| > 1$ (we neglect the displacement current)

$$\frac{\partial h_x^*}{\partial \zeta} = R_m^* e_y^*, \qquad -\frac{\partial h_y^*}{\partial \zeta} = R_m^* e_x^*, \qquad \frac{\partial e_x^*}{\partial \zeta} = -\frac{\partial h_y^*}{\partial \tau}, \qquad \frac{\partial e_y^*}{\partial \zeta} = \frac{\partial h_x^*}{\partial \tau} \tag{1.9}$$

with the initial conditions

$$h_x^* = h_y^* = 0 \quad \text{when } \tau = 0 \tag{1.10}$$

It will be assumed that when $|\zeta| \to \infty$ the electromagnetic fields remain bounded. Accordingly the problem reduces to solution of the systems (1.4) and (1.9) with the boundary conditions (1.8) and the initial conditions (1.7) and (1.10).

2. The general solution of the problem. Applying to (1.4) the Laplace transformation and introducing the notation

$$F(\zeta, p) = \int_0^\infty f(\zeta, \tau)\, e^{-p\tau}\, d\tau \tag{2.1}$$

we obtain for the region $|\zeta| < 1$ allowing for the initial conditions

(1.7)

$$pU_x = \frac{P}{p} + \frac{1}{R_m} H_x', \qquad pU_y = \frac{S}{R_m} H_y'$$
$$pH_x = \frac{1}{R_m} (H_x'' + \beta H_y'') + U_x', \qquad pH_y = \frac{1}{R_m} (H_y'' - \beta H_x'') + U_y' \tag{2.2}$$

Eliminating the quantities U_x and U_y leads to the following system relating the Laplace transforms of the induced magnetic fields

$$\begin{aligned} &\left(1 + \frac{S}{p}\right) H_x'' + \beta H_y'' - R_m p H_x = 0 \\ &\left(1 + \frac{S}{p}\right) H_y'' - \beta H_x'' - R_m p H_y = 0 \end{aligned} \qquad (|\zeta| < 1) \tag{2.3}$$

If we set

$$\varphi = h_x - i h_y \tag{2.4}$$

then the system (2.3) can be written in the form of a single equation

$$\left(1 + i\beta + \frac{S}{p}\right) \Phi'' - p R_m \Phi = 0 \tag{2.5}$$

Employing the Laplace transformation in the solution of Equations (1.9) with conditions (1.10) leads to the following system for the region $|\zeta| > 1$:

$$H_x^{*\prime} = R_m^* E_y^*, \quad pH_x^* = E_y^{*\prime}, \quad H_y^{*\prime} = - R_m^* E_x^*, \quad pH_y^* = -E_x^{*\prime} \tag{2.6}$$

Solving the equations

$$H_x^{*\prime\prime} - pR_m^* H_x^* = 0, \qquad H_y^{*\prime\prime} - pR_m^* H_y^* = 0 \tag{2.7}$$

which follow from (2.6), and taking into consideration the condition of boundedness when $|\zeta| \to \infty$, we have (here and in what follows the upper sign relates to the region $\zeta > 1$, whilst the lower sign refers to the region $\zeta < -1$)

$$H_x^* = M \exp(-|\zeta| \sqrt{R_m^* p}), \qquad H_y^* = N \exp(-|\zeta| \sqrt{R_m^* p}) \tag{2.8}$$

The relations so obtained enable us, without finding the intensity of the electromagnetic field in the channel walls, to construct the boundary conditions required for the solution of the fundamental equation (2.5). At the same time, applying the Laplace transformation to Formulas (1.6) and eliminating the quantities U_x and U_y with the help of Equations (2.2), we obtain

$$E_x = \frac{1}{R_m}\left[\beta H_x' - \left(1+\frac{S}{p}\right)H_y'\right], \quad E_y = \frac{1}{R_m}\left[\beta H_y' + \left(1+\frac{S}{p}\right)H_x'\right] + \frac{P}{p^2} \tag{2.9}$$

Now, expressing the electric fields in the walls in terms of the magnetic fields

$$E_x^* = \pm\sqrt{p / R_m^*}\, H_y^*, \qquad E_y^* = \mp\sqrt{p / R_m^*} H_x^* \tag{2.10}$$

and taking the Laplace transforms of (1.8), we obtain the required boundary conditions:

$$\left[\beta H_x' - \left(1+\frac{S}{p}\right)H_y'\right]_{\zeta=\pm 1} = \pm R_m \sqrt{p / R_m^*}\, H_y|_{\zeta=\pm 1} \tag{2.11}$$

$$\left[\beta H_y' + \left(1+\frac{S}{p}\right)H_x'\right]_{\zeta=\pm 1} + \frac{R_m P}{p^2} = \mp R_m \sqrt{p / R_m^*}\, H_x|_{\zeta=\pm 1}$$

which can be written in the following form

$$\left[\left(1+i\beta+\frac{S}{p}\right)\Phi' \pm R_m\sqrt{p / R_m^*}\,\Phi\right]_{\zeta=\pm 1} + \frac{PR_m}{p^2} = 0 \tag{2.12}$$

Accordingly, the problem consists in the solution of Equation (2.5) with the boundary conditions (2.12).

By virtue of the fact that the fields h_x and h_y are odd with respect to the coordinate ζ we obtain

$$\Phi = A \sinh\gamma\zeta, \qquad \gamma = \frac{p\sqrt{R_m}}{\sqrt{(1+i\beta)p+S}} \tag{2.13}$$

where the quantity A is found from the boundary conditions (2.12)

$$A = -\frac{1}{(1+i\rho+S/p)\,\gamma\cosh\gamma + R_m\sqrt{p / R_m^*}\sinh\gamma}\,\frac{PR_m}{p^2} \tag{2.14}$$

General formulas for the induced magnetic fields are easily obtained now with the help of the inversion theorems of Riemann and Mellin

$$h_x = \operatorname{Re}\varphi, \qquad h_y = -\operatorname{Im}\varphi \tag{2.15}$$

$$\varphi = -\frac{P}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\sinh\gamma\zeta}{p/\gamma\cosh\gamma + \sqrt{p / R_m^*}\sinh\gamma}\,\frac{\exp(p\tau)}{p^2}\,dp \tag{2.16}$$

Similar expressions for the velocities u_x and u_y can be found, starting from the following relation which arises from (2.2)

$$U_x - iU_y = \frac{P}{p^2} + \frac{S}{pR_m}\,\Phi' \tag{2.17}$$

Accordingly

$$u_x = P\tau + \operatorname{Re}\psi, \qquad u_y = -\operatorname{Im}\psi \tag{2.18}$$

where

$$\psi = -\frac{PS}{R_m}\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \frac{\gamma\cosh\gamma\zeta}{p\,/\,\gamma\cosh\gamma + \sqrt{p\,/\,R_m{}^*}\sinh\gamma}\,\frac{\exp(p\tau)}{p^3}\,dp \tag{2.19}$$

With this, the general solution of the problem can be regarded as complete.

The actual carrying out of the calculations putting the solution so obtained into tangible form turns out to be rather complicated, which is connected in particular with the multi-valued nature of the transformed functions and the mixed character of the spectrum of eigenvalues (see [4], where similar circumstances occurred in the flow of a viscous fluid with isotropic conductivity). Accordingly in what follows we shall restrict ourselves to the special case of ideally conducting channel walls.

3. Flow in a channel with ideally conducting walls. Setting $R_m{}^* = \infty$ in the formulas of the preceding section, we obtain the general solution of the problem in the following form:

$$\begin{aligned} h_x(\zeta,\tau) &= \operatorname{Re}\varphi(\zeta,\tau), \qquad h_y(\zeta,\tau) = -\operatorname{Im}\varphi(\zeta,\tau) \\ u_x(\zeta,\tau) &= P\tau + \operatorname{Re}\psi(\zeta,\tau), \qquad u_y(\zeta,\tau) = -\operatorname{Im}\psi(\zeta,\tau) \end{aligned} \tag{3.1}$$

where, as before, we have introduced the notation

$$\varphi = -\frac{P}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\frac{\gamma\sinh\gamma\zeta}{p^3\cosh\gamma}\exp(p\tau)\,dp, \qquad \psi = -\frac{PS}{2\pi i R_m}\int\limits_{b-i\infty}^{b+i\infty}\frac{\gamma^2\cosh\gamma\zeta}{p^4\cosh\gamma}\exp(p\tau)\,dp \tag{3.2}$$

Singular points of the integrands occur at the poles $p = 0$ and $p = p_n$: the latter are obtained by solution of the equation $\cosh\gamma = 0$ and give formulas ($n = 0, 1, 2, \ldots$)

$$p_{n_{1,2}} = \frac{\lambda_n{}^2}{2R_m}(1+i\beta)\left[-1 \mp \sqrt{1 - \frac{4R_mS}{\lambda_n{}^2(1+i\beta)^2}}\right], \qquad \lambda_n = \frac{2n+1}{2}\pi \tag{3.3}$$

Application of the theorem of residues leads to the solution of the problem in real form

$$h_x = h_x^\circ + 2PR_m \operatorname{Re} F_1, \qquad h_y = h_y^\circ - 2PR_m \operatorname{Im} F_1$$
$$u_x = u_x^\circ + 2PS \operatorname{Re} F_2, \qquad u_y = u_y^\circ - 2PS \operatorname{Im} F_2 \tag{3.4}$$

where we have introduced the following notation:

$$F_1(\zeta, \tau) = \frac{1}{1+i\beta} \sum_{n=0}^{\infty} \left[\frac{\exp(p_{n1}\tau)}{p_{n1}} - \frac{\exp(p_{n2}\tau)}{p_{n2}} \right] \frac{(-1)^n \sin \lambda_n \zeta}{\lambda_n^2 \sqrt{1 - 4R_m S / \lambda_n^2 (1+i\beta)^2}}$$
$$F_2(\zeta, \tau) = \frac{1}{1+i\beta} \sum_{n=0}^{\infty} \left[\frac{\exp(p_{n1}\tau)}{p_{n1}^2} - \frac{\exp(p_{n2}\tau)}{p_{n2}^2} \right] \frac{(-1)^n \cos \lambda_n \zeta}{\lambda_n \sqrt{1 - 4R_m S / \lambda_n^2 (1+i\beta)^2}} \tag{3.5}$$

and separated out the steady regime

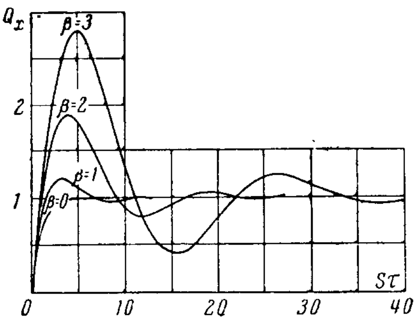

Fig. 2.

$$h_x^\circ = -\frac{R_m P}{S}\zeta, \quad h_y^\circ = 0, \quad u_x^\circ = \frac{P}{S}$$
$$u_y^\circ = -\frac{P}{S}\beta \tag{3.6}$$

Accordingly, in the case of ideally conducting channel walls there exists a steady regime, in which gas flows uniformly with velocity $v_x^\circ = P_x/H_0^2\sigma$ in the direction of the applied pressure gradient P_x, but also with velocity $v_y^\circ = -\beta v_x^\circ$ in the perpendicular direction. Moreover, there is flowing through the gas a constant current with density $j_y = -P_x/H_0$, and there is no electric field.

Setting $\beta = 0$ in the solution so obtained, we find the value of the fields and velocities for the case of isotropic conductivity*:

$$h_x = -\frac{R_m P}{S}\zeta + 2PR_m \operatorname{Re} \sum_{n=0}^{\infty} \left[\frac{\exp(q_{n1}\tau)}{q_{n1}} - \frac{\exp(q_{n2}\tau)}{q_{n2}} \right] \frac{(-1)^n \sin \lambda_n \zeta}{\lambda_n^2 \sqrt{1 - 4R_m S / \lambda_n^2}}$$
$$u_x = \frac{P}{S} + 2PS \operatorname{Re} \sum_{n=0}^{\infty} \left[\frac{\exp(q_{n1}\tau)}{q_{n1}^2} - \frac{\exp(q_{n2}\tau)}{q_{n2}^2} \right] \frac{(-1)^n \cos \lambda_n \zeta}{\lambda_n \sqrt{1 - 4R_m S / \lambda_n^2}} \tag{3.7}$$
$$h_y \equiv 0, \qquad u_y \equiv 0$$

where

* A similar problem is considered in [5].

$$q_{n_{1,2}} = \frac{\lambda_n^2}{2R_m}\left[-1 \mp \sqrt{1 - \frac{4R_m S}{\lambda_n^2}}\right] \tag{3.8}$$

From the formulas thus obtained it follows that when $SR_m < (\pi/4)^2$ all the q_n are real negative numbers, so that the transitional regime has an aperiodic nature. If, however, $SR_m > (\pi/4)^2$, then a certain finite number (N) of the numbers q_n are complex (with negative real parts) and the transition to the steady regime takes place by way of damped oscillations with frequencies

$$\frac{\lambda_n^2 v_0}{2R_m a}\sqrt{\frac{4R_m S}{\lambda_n^2} - 1} \qquad (0 \leqslant n \leqslant N-1)$$

It appears that the presence of anisotropic conductivity brings an essential change in the nature of the transition process, namely, the oscillatory process occurs for any arbitrarily small value of the magnetic Reynolds' number R_m.

In order to show this, let us consider the asymptotic expressions for the fields and velocities arising from formulas (3.4) and (3.5) when $R_m \ll 1$ ($\lambda = S/(1+\beta^2)$):

$$\begin{aligned} h_x &= -\frac{PR_m}{S}\xi\,[1 - \exp(-\lambda\tau)\cos\lambda\beta\tau] + O(R_m^2) \\ h_y &= -\frac{PR_m}{S}\zeta\exp(-\lambda\tau)\sin\lambda\beta\tau + O(R_m^2) \\ u_x &= \frac{P}{S}[1 - \exp(-\lambda\tau)(\cos\lambda\beta\tau - \beta\sin\lambda\beta\tau)] + O(R_m) \\ u_y &= -\frac{P}{S}[\beta - \exp(-\lambda\tau)(\sin\lambda\beta\tau + \beta\cos\lambda\beta\tau)] + O(R_m) \end{aligned} \tag{3.9}$$

It will at the same time be assumed that the magnetic interaction parameter S is a quantity of order unity. This is the case, for example, for sufficiently strong magnetic fields.

From the last formulas it is clear that when $R_m \ll 1$ and $\beta \neq 0$ (in the presence of anisotropic conductivity) the transition regime is accomplished in the form of oscillations with frequency $\lambda\beta$, whereas when $\beta = 0$ (in the absence of anisotropy) it has a purely aperiodic nature.

In conclusion we present graphs of the values of the transverse fluxes (Figs. 2 and 3) and the total electric currents flowing in the gas (Fig. 4), constructed with the help of Formulas (3.9) for various values of the time $S\tau = H_0^2\sigma/\rho t$ and the parameter of anisotropy β and referred to the quantities $2Pv_0\alpha/S$ and $PR_m H_0/2\pi S$, respectively.

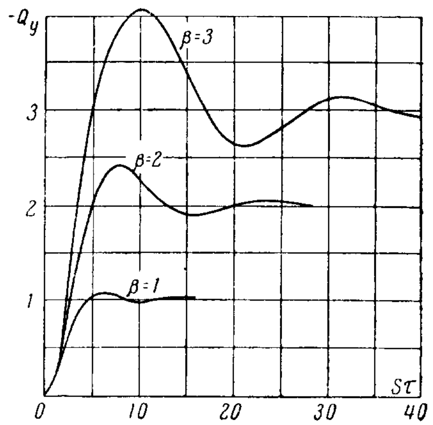

Fig. 3.

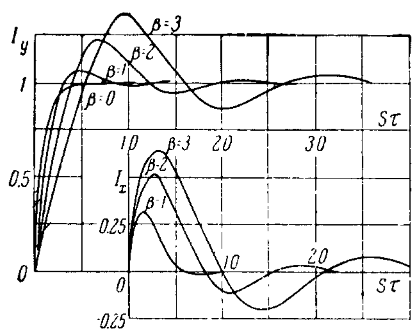

Fig. 4.

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